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CENTER FOR NAVAL ANALYSES
1401 Wilson Boulevard
Arlington, Virginia 22209

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CORRESPONDENCE RULES AND PATH INTEGRALS

Maurice M. Mizrahi

Center for Naval Analyses of
the University of Rochester
1401 Wilson Boulevard
Arlington, Va. 22209, USA

ABSTRACT

A path-integral representation is constructed for propagators corresponding to quantum Hamiltonian operators obtained from classical Hamiltonians by an arbitrary rule of correspondence. Each rule yields a unique way of defining the path integral in the context of a formalism which does not require a limiting process. This formalism is more reliable than the usual lattice definition in that all the expressions it entails are well-defined for computational purposes and it allows the explicit evaluation of large classes of path integrals. Direct substitution in the Schrödinger equation shows that there are no restrictions on the Hamiltonian operator. Examples are given.

I. INTRODUCTION

The purpose of this paper is to propose a solution to the following problem: Given an arbitrary classical Hamiltonian $H_c(p,q,t)$ and an arbitrary rule of correspondence which enables one to derive a quantum Hamiltonian operator $\underset{\sim}{H}(\underset{\sim}{P},\underset{\sim}{Q},t)$ (Hermitian or not) from H_c, find a path-integral representation for the propagator $K \equiv \langle q_b,t_b | q_a,t_a \rangle$ corresponding to $\underset{\sim}{H}$ which (1) takes proper account of the correspondence rule and (2) does not involve a limiting process (i.e. a "skeletonization of the path" or "time-slicing" technique) in its definition. The latter requirement purports to avoid the many ambiguities inherent in this process and to enable one to actually compute path integrals, rather than simply exhibit formal expressions.

In 1975 we showed[1], by time-slicing and Weyl transform techniques, that a formal path-integral expression in phase space can be written for the propagator, where the Weyl transform of the Hamiltonian operator takes the place of the classical Hamiltonian in the action functional. Subsequent work by Cohen[2] and Dowker[3] showed that formal path integrals can be obtained to accommodate any rule of correspondence. In all the foregoing papers, however, work stopped when a "formal" path integral was obtained, leaving open the problems of evaluation of the path integrals, substitution of the path-integral expression in the Schrödinger equation for verification, and justification of some possibly ambiguous limits inherent in the time-slicing approach. This paper will address the above problems by proposing an alternative approach, and supersedes Ref. 1.

The starting points are the general framework for path integration in phase space without limiting procedure introduced in Ref. 4 and Cohen's mappings between correspondence rules and ordinary functions[5]. It will be shown that an infinite series representing the propagator (where each term is a path integral) satisfies the Schrödinger equation with arbitrary $\underset{\sim}{H}$ and the boundary condition, provided a consistent well-defined algorithm is used to properly take the correspondence rule into account.

Only path integrals representations with respect to the free-particle measure will be considered here. The more general measures introduced in Ref. 4, which allow a semiclassical expansion of the propagator, will be treated elsewhere.[6] We work in one dimension to simplify the discussion, although the results can be readily generalized to n dimensions. All integrals are over $\mathbb{R}^s$ for suitable s, unless otherwise specified.

II. THE PATH-INTEGRAL FORMALISM

The formalism for constructing phase-space path integrals without resorting to a limiting procedure with ambiguous epsilonics was introduced in Ref. 4 and only a brief description will be given here[7]. It consists of defining what plays the role of a measure in phase space by its Fourier transform, which is a simple closed-form expression. For example, the normalized free-particle measure w(p,q) in phase space $\mathcal{P}$, corresponding to

$$dw(p,q) \sim \frac{1}{K_0}\left[\frac{dp\,dq}{2\pi\hbar}\right]\exp\left\{\frac{i}{\hbar}\int_{t_a}^{t_b}\left[p(t)\dot{q}(t) - \frac{p^2(t)}{2m}\right]dt\right\}, \tag{1}$$

can be shown[4] to have, as its Fourier transform:

$$\mathcal{F}w(\mu,\nu) = \exp\Big\{ -i\langle\mu,\bar{q}\rangle - i\langle\nu,\bar{p}\rangle - \frac{i\hbar}{2}\int_T\int_T G_{ab}(t,t')\,d\mu(t)\,d\mu(t')$$

$$-i\hbar\int_T\int_T \bar{G}(t,t')\,d\mu(t)\,d\nu(t') - \frac{i\hbar}{2}\int_T\int_T G_p(t,t')\,d\nu(t)\,d\nu(t')\Big\}, \tag{2}$$

where:

(1) $\mathcal{P} \equiv \{[p(t),q(t)] \text{ on } T \equiv [t_a,t_b] \mid q(t_a)=q_a,\ q(t_b)=q_b,\ p(t) \text{ unrestricted}\}$, (3)

(2) $$K_0 = \left(\frac{m}{2\pi i\hbar T}\right)^{1/2} \exp\left[\frac{im(q_b-q_a)^2}{2\hbar T}\right] \tag{4}$$

is the free-particle propagator $(T \equiv t_b - t_a)$,

(3) $[\bar{q}(t),\bar{p}(t)]$ is the average free-particle path:

$$\bar{q}(t) = [q_b(t-t_a) + q_a(t_b-t)]/T \tag{5}$$

$$\bar{p}(t) = m(q_b-q_a)/T \tag{6}$$

(4) The G functions are the free-particle covariances:

$$G_{ab}(t,t') = [(t'-t_a)(t_b-t)Y(t-t') + (t-t_a)(t_b-t')Y(t'-t)]/mT \tag{7}$$

$$\bar{G}(t,t') = [(t_b-t)Y(t-t') - (t-t_a)Y(t'-t)]/T \tag{8}$$

$$G_p(t,t') = -m/T \tag{9}$$

[$Y(x)$ being 1 when $x > 0$ and 0 otherwise],

(5) μ and ν are elements of $\mathcal{M}$, the space of bounded measures on the time interval T; $\langle\mu,q\rangle \equiv \int_T q(t)\,d\mu(t)$ if μ is induced by a function, $\langle\delta_t,q\rangle \equiv q(t)$ if μ is δ_t, the "delta-function" measure at t.

The most general Gaussian measure, which absorbs all the quadratic terms [i.e. not only $p^2/2m$ but $g(t)p^2/2m + f(t)q^2/2 + k(t)pq$] in either the Hamiltonian or the action functional expanded about the classical path, was constructed in Ref. 4, but the free-particle measure will be sufficient for our purpose here. w is a true phase-space measure: it does not entail performing separate, successive path integrals in configuration space and momentum space.

This definition enables one to carry out path integrals of cylindrical functionals, i.e. functionals which depend on only a finite number of terms of the form $\langle \mu, q \rangle$ or $\langle \nu, p \rangle$, by converting them into finite-dimensional ordinary integrals. The result is the following fundamental integral [in a form slightly different from that given in Ref.4, Eq. (97)]:

$$\int_{\mathcal{P}} F(\langle \mu_1, q\rangle, \ldots, \langle \mu_n, q\rangle, \langle \nu_1, p\rangle, \ldots, \langle \nu_m, p\rangle)\, dw(p,q)$$

$$= \int_{\mathbb{R}^{n+m}} \frac{F(y)\,dy}{\left[(2\pi i \hbar)^{m+n} \det A\right]^{1/2}} \exp\left\{ \frac{i}{2\hbar} \sum_{i,j=1}^{n+m} (A^{-1})_{ij} (y_i - a_i)(y_j - a_j) \right\} \tag{10}$$

where

$$y \equiv (u_1, \ldots, u_n, v_1, \ldots, v_m) \tag{11}$$

$$dy \equiv du_1 \ldots du_n\, dv_1 \ldots dv_m \tag{12}$$

$$a \equiv (\langle \mu_1, \bar{q}\rangle, \ldots, \langle \mu_n, \bar{q}\rangle, \langle \nu_1, \bar{p}\rangle, \ldots ; \langle \nu_m, \bar{p}\rangle) \tag{13}$$

$$A \equiv \begin{pmatrix} W & C \\ \tilde{C} & V \end{pmatrix} \qquad [(n+m)\times(n+m)] \qquad (14)$$

$$W_{ij} \equiv \int_T\int_T G_{ab}(t,t')\, d\mu_i(t)\, d\mu_j(t') \qquad (n\times n) \qquad (15)$$

$$C_{ij} \equiv \tilde{C}_{ji} \equiv \int_T\int_T \bar{G}(t,t')\, d\mu_i(t)\, d\nu_j(t') \qquad (n\times m) \qquad (16)$$

$$V_{ij} \equiv \int_T\int_T G_p(t,t')\, d\nu_i(t)\, d\nu_j(t') \qquad (m\times m) \qquad (17)$$

In the above formula, w is not restricted to being the free-particle measure, but can be the most general quadratic measure mentioned earlier. Of particular interest are an expression for the Fourier transform of w:

$$\int_{\mathcal{P}} \left\{ \exp\left[-i\langle \mu, q\rangle - i\langle \nu, p\rangle \right] \right\} dw(p,q) = \mathcal{F}w(\mu,\nu) \qquad (18)$$

and the special case where $F(x_1,\ldots,x_k) = x_1\ldots x_k$, which yields the generalized moments formula in phase space [an extension of the one given in Ref. 8, Eq. (65), for configuration space] :

$$\int_{\mathcal{P}} \langle \mu_1, q\rangle \cdots \langle \mu_n, q\rangle \langle \nu_1, p\rangle \cdots \langle \nu_m, p\rangle \, dw(p,q)$$

$$= i^{n+m} \mathcal{H}_{n+m}\left(-\frac{ic_1}{2}, \ldots, -\frac{ic_{n+m}}{2} \right) \qquad (19)$$

where

$$c_k \equiv \begin{cases} \langle \mu_k, \bar{q} \rangle & \text{for } k = 1, \ldots, n \\ \langle \nu_k, \bar{p} \rangle & \text{for } k = n+1, \ldots, n+m \end{cases} \tag{20}$$

and $\mathcal{H}_{n+m}$ is the generalized Hermite polynomial of order n+m and matrix $i\hbar A/2$, defined in Refs. 8 and 9[10]. The examples shown below, for the case where all μs and νs are δ functions, reveal the general pattern:

$$\int_{\mathcal{P}} q(t)\, dw(p,q) = \bar{q}(t) \tag{21}$$

$$\int_{\mathcal{P}} q(t)\, p(t')\, dw(p,q) = \bar{q}(t)\,\bar{p}(t') + i\hbar\,\bar{G}(t,t') \tag{22}$$

$$\int_{\mathcal{P}} q(t)\, q(t')\, dw(p,q) = \bar{q}(t)\,\bar{q}(t') + i\hbar\, G_{ab}(t,t') \tag{23}$$

$$\int_{\mathcal{P}} p(t)\, p(t')\, dw(p,q) = \bar{p}(t)\,\bar{p}(t') + i\hbar\, G_{p}(t,t') \tag{24}$$

$$\begin{aligned} \int_{\mathcal{P}} q(t_1)\, q(t_2)\, p(t_3)\, dw(p,q) = {} & \bar{q}(t_1)\,\bar{q}(t_2)\,\bar{p}(t_3) \\ & + i\hbar\,\bar{G}(t_2,t_3)\,\bar{q}(t_1) + i\hbar\,\bar{G}(t_1,t_3)\,\bar{q}(t_2) \\ & + i\hbar\, G_{ab}(t_1,t_2)\,\bar{p}(t_3) \end{aligned} \tag{25}$$

$$\int_{\mathcal{P}} q(t_1)q(t_2)p(t_3)p(t_4)\,dw(p,q) = \bar{q}(t_1)\bar{q}(t_2)\bar{p}(t_3)\bar{p}(t_4)$$

$$+ i\hbar\,\bar{G}(t_1,t_4)\,\bar{q}(t_2)\bar{p}(t_3) + i\hbar\,\bar{G}(t_2,t_4)\,\bar{q}(t_1)\bar{p}(t_3)$$

$$+ i\hbar\,G_p(t_3,t_4)\,\bar{q}(t_1)\,\bar{q}(t_2) + i\hbar\,G_{ab}(t_1,t_2)\,\bar{p}(t_3)\,\bar{p}(t_4)$$

$$+ i\hbar\,\bar{G}(t_2,t_3)\,\bar{q}(t_1)\bar{p}(t_4) + i\hbar\,\bar{G}(t_1,t_3)\,\bar{q}(t_2)\bar{p}(t_4)$$

$$+ (i\hbar)^2\,G_{ab}(t_1,t_2)\,G_p(t_3,t_4) + (i\hbar)^2\,\bar{G}(t_2,t_3)\,\bar{G}(t_1,t_4)$$

$$+ (i\hbar)^2\,\bar{G}(t_1,t_3)\,\bar{G}(t_2,t_4) \qquad (26)$$

It will be noted that the pq correlation function, $\bar{G}(t,t')$, is discontinuous across the diagonal $t = t'$, and its jump there is of magnitude 1:

$$\left[\lim_{(t-t')\to 0^+} - \lim_{(t'-t)\to 0^+}\right]\bar{G}(t,t') = 1 \qquad (27)$$

[see, e.g., (8) for the free-particle case]. Thus, $\mathcal{F}w(\delta_t,\delta_t)$ is not defined. This indefiniteness, which occurs in a natural manner as one builds the measure[4], stems from the non-commutativity of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$, and will give us the flexibility we need to take various correspondence rules into account.

III. THE CORRESPONDENCE RULES

We will consider the most general quantum Hamiltonian operator $\underset{\sim}{H}(\underset{\sim}{P},\underset{\sim}{Q},t)$ which can be derived from a classical Hamiltonian $H_c(p,q,t)$. The form most convenient for our purposes is that given by Cohen[5]:

$$\underset{\sim}{H} = (2\pi\hbar)^{-2}\int dp\,dq\,du\,dv\; F(u,v,\hbar)\, H_c(p,q,t) \times \exp\left\{(i/\hbar)\left[(q-\underset{\sim}{Q})u + (p-\underset{\sim}{P})v\right]\right\}, \tag{28}$$

where $F(u,v,\hbar)$ is the transformation function[11]. Each F uniquely determines the correspondence rule. For instance, $F = 1$, $F = \cos(uv/2\hbar)$ and $F = (uv/2\hbar)^{-1}\sin(uv/2\hbar)$ give the Weyl, symmetrized, and Born-Jordan ordering schemes, respectively. For real H_c, $\underset{\sim}{H}$ is Hermitian iff $F^*(-u,-v,\hbar) = F(u,v,\hbar)$, a condition we will not need. Also, we must have $F(0,v,\hbar) = F(u,0,\hbar) = 1$ to insure that $f(\underset{\sim}{P})$ and $g(\underset{\sim}{Q})$ correspond to $f(p)$ and $g(q)$. The transform can be inverted to yield

$$H_c(p,q,t) = (2\pi\hbar)^{-1}\int dq'\,dp'\,dq''\, \exp\left\{(i/\hbar)\left[q'p + p'(q-q'')\right]\right\} \times F^{-1}(q',p',\hbar)\,\langle q'' - \tfrac{1}{2}q'|\,\underset{\sim}{H}\,|q'' + \tfrac{1}{2}q'\rangle, \tag{29}$$

and finally, the transformation function F can be inferred from the knowledge of the Hamiltonian H_c and its transform $\underset{\sim}{H}$:[12]

$$F(u,v,\hbar)\int dp\,dq\,H_c(p,q,t)\exp\left[i(qu+pv)/\hbar\right]$$

$$=(2\pi\hbar)\,\mathrm{tr}\left(e^{i(\underset{\sim}{Q}u+\underset{\sim}{P}v)/\hbar}\,\underset{\sim}{H}\right). \quad (30)$$

Note that the F which relates a given $\underset{\sim}{H}$ with a given H_c is usually not unique. For example, for $\underset{\sim}{H} = f(\underset{\sim}{Q})$ and $H_c = f(q)$, (30) gives:

$$F(u,v,\hbar)\,\delta(v)\int dq\,f(q)\,e^{iqu/\hbar} = \int dq\,f(q)\,e^{iqu/\hbar}. \quad (31)$$

Thus, any F such that $F(u,0,\hbar) = 1$ will do. Similarly, one finds that pq is mapped into $(\underset{\sim}{P}\underset{\sim}{Q} + \underset{\sim}{Q}\underset{\sim}{P})/2$ by any $F = F(uv/\hbar)$ such that $F(0) = 1$ and $F'(0) = 0$.

It will be more convenient in certain cases to put $\underset{\sim}{H}$ in (28) in its normal-ordered form ($\underset{\sim}{Q}$ before $\underset{\sim}{P}$). For this, one uses the Baker-Campbell-Hausdorff formula, $e^{\underset{\sim}{A}+\underset{\sim}{B}} = e^{\underset{\sim}{A}}e^{\underset{\sim}{B}}e^{-[\underset{\sim}{A},\underset{\sim}{B}]/2}$, valid for all $\underset{\sim}{A}$ and $\underset{\sim}{B}$ which commute with $[\underset{\sim}{A},\underset{\sim}{B}]$. It gives:

$$e^{-i(\underset{\sim}{Q}u+\underset{\sim}{P}v)/\hbar} = e^{-i\underset{\sim}{Q}u/\hbar}\,e^{-i\underset{\sim}{P}v/\hbar}\,e^{iuv/2\hbar}. \quad (32)$$

Since $\underset{\sim}{H}$ will be applied to functions of the endpoint q_b, $\underset{\sim}{Q}$ is represented by q_b and $\underset{\sim}{P}$ by $-i\hbar\,\partial/\partial q_b$. Thus, the most general operator $\underset{\sim}{H}$ derived from H_c is, in its normal-ordered form,

$$\underset{\sim}{H} = (2\pi\hbar)^{-2}\int dp\,dq\,du\,dv\,F(u,v,\hbar)\,H_c(p,q,t)$$

$$\times\exp\left[\frac{iu}{\hbar}\left(q-q_b+\frac{v}{2}\right)\right]\exp\left(\frac{ipv}{\hbar}\right)\exp\left(-v\frac{\partial}{\partial q_b}\right). \quad (33)$$

Consequently, when $\underset{\sim}{H}$ is applied to $f(q_b)$, the result is the right-hand side of (33) with $\exp(-v\,\partial/\partial q_b)$ replaced by $f(q_b - v)$.

IV. THE PROPAGATOR BY PATH INTEGRALS, FOR ARBITRARY HAMILTONIANS

With the two foregoing tools in hand, we can proceed to write a path-integral representation for the propagator corresponding to an arbitary Hamiltonian operator. First, we write the latter in the form:

$$\underset{\sim}{H} = \frac{\underset{\sim}{P}^2}{2m} + k\,\underset{\sim}{H}_1(\underset{\sim}{P},\underset{\sim}{Q},t). \tag{34}$$

This is done to split off the free-particle part so we can use the free-particle measure $w(p,q)$. Although the propagator will be expressed as a power-series in k, our aim is <u>not</u> a perturbation expansion. Rather, it is the manner in which the ordering of the factors in $\underset{\sim}{H}_1$ is taken into account in evaluating the path integrals, so that the propagator K thereby obtained satisfies the Schrödinger equation associated with $\underset{\sim}{H}$ to all orders in k. Thus, k should be regarded as simply a "bookkeeping" parameter. Second, we define a function $H_{no}(p,q,t)$, obtained by normal-ordering the operator $\underset{\sim}{H}_1$ and then replacing $\underset{\sim}{P}$ by p and $\underset{\sim}{Q}$ by q. For example (since $\underset{\sim}{Q}\underset{\sim}{P}-\underset{\sim}{P}\underset{\sim}{Q} = i\hbar$):

$$H_c = p^2/2m + kpq^2 \tag{35}$$

$$\underset{\sim}{H} = \underset{\sim}{P}^2/2m + k\,\underset{\sim}{Q}\underset{\sim}{P}\underset{\sim}{Q} = \underset{\sim}{P}^2/2m + k(\underset{\sim}{Q}^2\underset{\sim}{P} - i\hbar\underset{\sim}{Q}) \tag{36}$$

$$H_{no} = q^2 p - i\hbar q. \tag{37}$$

Our main result is expressed in the following theorem.

<u>Theorem</u>

The propagator $K \equiv \langle q_b,t_b | q_a,t_a \rangle$, or probability amplitude that a particle at position q_a at time t_a will be at position q_b at time t_b, for an arbitrary Hamiltonian operator $\underset{\sim}{H}$ [which we write as $\underset{\sim}{P}^2/2m + k\underset{\sim}{H}_1(\underset{\sim}{P},\underset{\sim}{Q},t)$, $\underset{\sim}{H}_1$

being arbitrary], can be written as a phase space path integral as follows:

$$K \doteq K_o \int_{\mathcal{P}} dw(p,q) \exp\left\{ -\frac{ik}{\hbar} \int_T H_{no}[p(t),q(t),t]\,dt \right\} \tag{38}$$

$$\doteq K_o \left\{ 1 + \sum_{j=1}^{\infty} \left(-\frac{ik}{\hbar}\right)^j \frac{1}{j!} \int_{\mathcal{P}} dw(p,q) \left[\int_T H_{no}(p(t),q(t),t)\,dt \right]^j \right\} \tag{39}$$

$$\doteq K_o \left\{ 1 + \sum_{j=1}^{\infty} \left(-\frac{ik}{\hbar}\right)^j \frac{1}{j!} \int_{\mathcal{P}} dw(p,q) \int_T dt_1\, H_{no}[p(t_1),q(t_1),t_1] \cdots \int_T dt_j\, H_{no}[p(t_j),q(t_j),t_j] \right\} \tag{40}$$

$$\equiv K_o \left\{ 1 + \sum_{j=1}^{\infty} \left(-\frac{ik}{\hbar}\right)^j \frac{1}{j!} \int_{T^j} dt_1 \cdots dt_j \times \lim_{(t_1-t_1')\to 0^+} \cdots \lim_{(t_j-t_j')\to 0^+} \int_{\mathcal{P}} dw(p,q) \times H_{no}[p(t_1'),q(t_1),t_1] \cdots H_{no}[p(t_j'),q(t_j),t_j] \right\}, \tag{41}$$

where

(1) K_o is the free-particle propagator (4), $w(p,q)$ is the free-particle measure defined by (1) and (2)ff, and $H_{no}(p,q,t)$ is a function obtained by

normal-ordering $\underset{\sim}{H}_1$ (i.e. $\underset{\sim}{Q}$ before $\underset{\sim}{P}$), then replacing $\underset{\sim}{Q}$ by q and $\underset{\sim}{P}$ by p;

(2) The dot over the equal sign in the first three equations, (38)-(40), denotes a formal, as yet undefined, relation. The equations are undefined because as soon as one interchanges the path integral over $\mathcal{P}$ and the time integral over T^j, one is faced with the path integral $\int_{\mathcal{P}} dw(p,q)H_{no}(p(t),q(t),t)$. Since the integrand contains terms coupling p and q at the same time t, this path integral is undefined bacause, as explained earlier, the pq correlation function, $\widetilde{G}(t,t')$ in (8), is undefined at $t = t'$. However, if we replace $\int_{\mathcal{P}} dw(p,q)H_{no}(p(t),q(t),t)$ by $\lim_{(t-t')\to 0^+} \int_{\mathcal{P}} dw(p,q)H_{no}(p(t'),q(t),t)$, the resulting expression is well-defined. This is done in the last equation, (41), which gives an unambiguous path-integral representation of K directly tied to the ordering of the factors in the quantum operator $\underset{\sim}{H}_1$;

(3) It is not necessary that the H function in (38)-(41) be H_{no}. However, if another is chosen, the "time" limits in (41) will be different. There is a close connection between the function chosen to replace H_{no} and the type of time limits which will give a well-defined, correct expression for the propagator K. This will be proved and discussed in lemma 1 below;

(4) The propagator (41) satisfies the Schrödinger equation and the boundary condition:

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_b^2} + k\,\underset{\sim}{H}_1\left(-i\hbar\frac{\partial}{\partial q_b}, q_b, t_b\right) - i\hbar\frac{\partial}{\partial t_b}\right]\langle q_b,t_b|q_a,t_a\rangle = 0 \qquad (42a)$$

$$\langle q_b, t_b | q_a, t_b \rangle = \delta(q_b - q_a), \qquad (42b)$$

to all orders in k;

(5) The path integral in (41) can be evaluated in case a perturbation series in k is sought. The result, proved later in the paper, can be expressed in terms of only the classical Hamiltonian H_1 and the correspondence function F. It is:

$$K = K_0 \left[1 + \sum_{j=1}^{\infty} k^j \alpha_j \right], \qquad (43)$$

where α_j is displayed in (70) and (71) below.

The proof of the theorem will consist in showing, by recurrence, that (41) satisfies (42) by using both (28) and (33) to relate the classical and quantum Hamiltonians. First, we give a *simple illustration of the theorem.*

Example

Calculate, to first order in k, the propagator corresponding to the following Hamiltonian:

$$\underset{\sim}{H} = \underset{\sim}{P}^2/2m + k\left[\alpha \underset{\sim}{P}\underset{\sim}{Q}^2 + \beta \underset{\sim}{Q}\underset{\sim}{P}\underset{\sim}{Q} + (1-\alpha-\beta)\underset{\sim}{Q}^2\underset{\sim}{P}\right], \qquad (44)$$

(which represents all the possible $\underset{\sim}{H}$s corresponding to $H_c = p^2/2m + kpq^2$).

Answer. The normal-ordered H is:

$$\underset{\sim}{H} = \underset{\sim}{P}^2/2m + k\left[\underset{\sim}{Q}^2\underset{\sim}{P} - i\hbar(\beta + 2\alpha)\underset{\sim}{Q}\right]. \qquad (45)$$

Using (41), we have:

$$K = K_0 \left\{ 1 - \frac{ik}{\hbar} \int_T dt \left[\lim_{t-t'\to 0^+} \int_{\mathcal{P}} q^2(t)\, p(t')\, dw(p,q) \right.\right.$$
$$\left.\left. - i\hbar(\beta+2\alpha) \int_{\mathcal{P}} q(t)\, dw(p,q) \right] \right\} \qquad (46)$$

$$= K_0 \left[1 - \frac{ikm}{3\hbar} (q_b^3 - q_a^3) + \frac{kT}{2} (q_b + q_a)(1-\beta-2\alpha) \right], \qquad (47)$$

where we have used (25), (21), and (5)-(9). It can be directly verified that (47) satisfies the Schrödinger equation to first order in k, i.e. that

$$\left\{ -(\hbar^2/2m) \frac{\partial^2}{\partial q_b^2} - i\hbar k \left[q_b^2 \frac{\partial}{\partial q_b} + (\beta - 2\alpha) q_b \right] - i\hbar \frac{\partial}{\partial t_b} \right\} K = O(k^2) \qquad (48)$$

together with $\lim_{t_b \to t_a} K = \delta(q_b - q_a)$.

Note that we could have obtained the same result without normal-ordering first, as stated earlier. If we consider the function obtained by replacing $\underset{\sim}{Q}$ and $\underset{\sim}{P}$ by q and p in (44), order the times in the sequence suggested by (44), then take successive coincidence limits, we get:

$$K = K_0 \left\{ 1 - \frac{ik}{\hbar} \int_T dt \left[\lim_{t-t'\to 0^+} \int_{\mathcal{P}} p(t)\, q^2(t')\, dw(p,q) \right.\right.$$
$$+ \beta \lim_{t-t'\to 0^+} \lim_{t'-t''\to 0^+} \int_{\mathcal{P}} q(t)\, p(t')\, q(t'')\, dw(p,q)$$
$$\left.\left. + (1-\alpha-\beta) \lim_{t-t'\to 0^+} \int_{\mathcal{P}} q^2(t)\, p(t')\, dw(p,q) \right] \right\}, \qquad (49)$$

which also yields the correct result (47). The general proof of this flexibility is found in lemma 1 below.

Proof of theorem

Lemma 1. For all phase-space functionals F[q,p] which do not contain the path (p,q) evaluated at either t or t', we have:

$$\lim_{t-t'\to 0^+} \int_{\mathcal{P}} F[q,p]\,[q(t)p(t')-q(t')p(t)-i\hbar]\,dw(p,q) = 0\,. \tag{50}$$

Proof. Consider the measure

$$dw_{tt'}(p,q) \equiv [q(t)p(t')-q(t')p(t)-i\hbar]\,dw(p,q)\,. \tag{51}$$

Its Fourier transform is

$$\mathcal{F}w_{tt'}(\mu,\nu) = \int_{\mathcal{P}} e^{-i\langle\mu,q\rangle - i\langle\nu,p\rangle}\,dw_{tt'}(p,q)\,. \tag{52}$$

To evaluate it, we proceed as follows:

$$\int_{\mathcal{P}} q(t)p(t')\,e^{-i\langle\mu,q\rangle - i\langle\nu,p\rangle}\,dw(p,q)$$

$$= -\frac{\partial^2}{\partial\lambda\partial\sigma}\int_{\mathcal{P}} e^{-i\langle\mu+\lambda\delta_t,q\rangle - i\langle\nu+\sigma\delta_{t'},p\rangle}\,dw(p,q)\Big|_{\lambda=\sigma=0}$$

$$= -\frac{\partial^2}{\partial\lambda\partial\sigma}\,\mathcal{F}w(\mu+\lambda\delta_t,\ \nu+\sigma\delta_{t'})\Big|_{\lambda=\sigma=0}$$

$$= \mathcal{F}w(\mu,\nu)\Big[\frac{i\hbar}{2}G_{ab}(t,t') + \frac{i\hbar}{2}G_p(t,t') + i\hbar\bar{G}(t,t') + \xi(\mu,\nu;t,t')\Big], \tag{53}$$

where

$$\xi(\mu,\nu;t,t') \equiv \Big[i\bar{q}(t) + \frac{i\hbar}{2}\int_T G_{ab}(t,s')d\mu(s') + \frac{i\hbar}{2}\int_T G_p(t,s')d\nu(s') + i\hbar\int_T \bar{G}(t,s')d\nu(s')\Big]$$
$$\times\Big[i\bar{p}(t') + \frac{i\hbar}{2}\int_T G_{ab}(s,t')d\mu(s) + \frac{i\hbar}{2}\int_T G_p(s,t')d\nu(s) + i\hbar\int_T \bar{G}(s,t')d\mu(s)\Big] \tag{54}$$

and (2) was used. Therefore,

$$\mathcal{F}w_{tt'}(\mu,\nu) = \mathcal{F}w(\mu,\nu)\Big\{-i\hbar + \frac{i\hbar}{2}\big[G_{ab}(t,t') - G_{ab}(t',t)\big] + \frac{i\hbar}{2}\big[G_p(t,t') - G_p(t',t)\big] + i\hbar\big[\bar{G}(t,t') - \bar{G}(t',t)\big] + \xi(\mu,\nu;t,t') - \xi(\mu,\nu);t',t)\Big\}. \tag{55}$$

Since G_{ab}, G_p, and ξ are continuous across the diagonal $t = t'$ (provided μ and ν are different from δ_t and $\delta_{t'}$), and since $\bar{G}$ has a jump of magnitude 1 there [Eq. (27)], we conclude that

$$\lim_{t-t'\to 0^+} \mathcal{F}w_{tt'}(\mu,\nu) = 0. \tag{56}$$

Consequently, the measure $w_{tt'}(p,q)$ is effectively the zero measure in the limit $(t-t')\to 0^+$, provided (1) this limit is taken after a path integral with respect to $w_{tt'}(p,q)$ is performed [(50) is obviously false if the limit is taken before the path integral is done], and (2) the integrand does not contain p or q evaluated at t or t' [if it does, then $\xi(\mu,\nu;t,t')$ is no longer continuous across the diagonal $t = t'$]. Q.E.D.

This lemma insures that the various path integrals obtained by changing the form of the given Hamiltonian operator (by repeated use of the commutation relation $\underset{\sim}{Q}\underset{\sim}{P}-\underset{\sim}{P}\underset{\sim}{Q} = i\hbar$) will all yield the same result. This was illustrated in the example above where the two path integrals (46) and (49), corresponding to the same Hamiltonian operator written in two different forms (45) and (44), gave the same correct answer.

Therefore, it is sufficient to prove the theorem for the normal-ordered form of $\underset{\sim}{H}_1$, i.e. with H_{no}.

Lemma 2.

$$\left(\int_{t_a}^{t_b} f(t)\,dt\right)^n = \int_{T^n} dt_1 \ldots dt_n\, f(t_1)\ldots f(t_n) \tag{57}$$

$$= n! \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_1} dt_2 \cdots \int_{t_a}^{t_{n-2}} dt_{n-1} \int_{t_a}^{t_{n-1}} dt_n\, f(t_1)\ldots f(t_n).$$

Proof. The lemma is true for n = 1. Assume it is true for n = k - 1. Consider $F_k(s) \equiv \left(\int_{t_a}^{s} f(x)\,dx\right)^k$. Then

$$\dot{F}_k(s) = k\,F_{k-1}(s)f(s) = k\,f(s)(k-1)!$$
$$\times \int_{t_a}^{s} dt_1 \int_{t_a}^{t_1} dt_2 \cdots \int_{t_a}^{t_{k-2}} dt_{k-1}\, f(t_1)\cdots f(t_{k-1}) . \qquad (58)$$

Integrating from t_a to t_b with respect to s gives

$$F_k(t_b) = k! \int_{t_a}^{t_b} ds \int_{t_a}^{s} dt_1 \int_{t_a}^{t_1} dt_2 \cdots \int_{t_a}^{t_{k-2}} dt_{k-1}\, f(t_1) \cdots$$
$$f(t_{k-1}) f(s). \qquad (59)$$

Changing variables: $s = t_1$, $t_1 = t_2$, ..., $t_{k-1} = t_k$, proves that the formula is true for n = k. Therefore the formula is proved true by recurrence. Q.E.D.

Proof to first order in k. The theorem will be proved by recurrence. Thus, we must first prove that (41) satisfies the Schrödinger equation (42) to first order in k. The propagator to first order in k is:

$$K = K_0 \left[1 - \frac{ik}{\hbar} \int_T dt \lim_{t-t' \to 0^+} \int_{\mathcal{P}} d\omega(p,q)\, H_{no}\left[p(t', q(t), t\right]\right]. \qquad (60)$$

Using the correspondence rule (28), along with (32) to put $\underset{\sim}{H}_1$ in normal-ordered form, we can write:

$$H_{no}[p(t'), q(t), t] = (2\pi\hbar)^{-2} \int dp\,dq\,du\,dv\; F(u,v,\hbar)\, H_1(p,q,t)$$
$$\times \exp\left\{ (i/\hbar)(qu + pv + uv/2) - iuq(t)/\hbar - ivp(t')/\hbar \right\} . \qquad (61)$$

Substituting (61) in (60) reveals that the path integral is a particularly simple one, namely the Fourier transform of w at $(u\delta_t/\hbar\ ,v\delta_{t'}/\hbar)$. It can be evaluated using (2) and (5)-(9), and the result, after the limit, is:

$$K = K_0 \left[1 + k\alpha_1 \right] , \qquad (62)$$

where

$$\alpha_1 \equiv -\frac{i}{\hbar(2\pi\hbar)^2} \int dp\, dq\, du\, dv\, F(u,v,\hbar) \int_T dt\, H_1(p,q,t)$$
$$\times \exp\left\{ (i/\hbar)(qu + pv + uv/2) - (i/\hbar T)\left[uq_b(t-t_a) \right.\right.$$
$$+ uq_a(t_b - t) + mv(q_b - q_a) + u^2(t-t_a)(t_b-t)/2m$$
$$\left.\left. + uv(t_b - t) - mv^2/2 \right] \right\} . \qquad (63)$$

We must show that

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_b^2} + k\,\underset{\sim}{H}_1\!\left(q_b, -i\hbar\frac{\partial}{\partial q_b}, t_b \right) - i\hbar\frac{\partial}{\partial t_b} \right]$$
$$\times \left[K_0 (1 + k\alpha_1) \right] = O(k^2) . \qquad (64)$$

Since K_0 satisfies the free-particle Schrödinger equation, we must simply show that the coefficient of k is zero, i.e. that

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_b^2} - i\hbar\frac{\partial}{\partial t_b} \right](K_0\alpha_1) + \underset{\sim}{H}_1 K_0 = 0 . \qquad (65)$$

For $\underset{\sim}{H}_1$ we will use (33), its normal-ordered form in terms of the classical function H_1. Thus,

$$\underset{\sim}{H}_1 K_0 = (2\pi\hbar)^{-2} \int dp\, dq\, du\, dv\, F(u,v,\hbar)\, H_1(p,q,t_b)$$
$$\times \left\{ \exp\left[\frac{iu}{\hbar}\left(q - q_b + \frac{v}{2}\right) + \frac{ipv}{\hbar} \right] \right\} \left(\frac{m}{2\pi i\hbar T} \right)^{1/2}$$
$$\times \exp\left[\frac{im}{2\hbar T}(q_b - v - q_a)^2 \right]. \qquad (66)$$

The remainder of the proof is tedious and straightforward. Differentiations with respect to q_b and t_b are performed under the integral sign, assuming interchangeability, using Leibnitz's rule, $(\partial/\partial t_b)\int_{t_a}^{t_b} \varphi(t,t_b)\,dt = \varphi(t_b,t_b) + \int_{t_a}^{t_b}[\partial\varphi(t,t_b)/\partial t_b]\,dt$, where needed. The $\underset{\sim}{H}_1 K_0$ term cancels the term equivalent to $\varphi(t_b,t_b)$. Upon collecting terms, the integrand vanishes, and the theorem is established to first order in k for arbitrary Hamiltonian operators.

Proof to any order in k. Using lemma 2 [Eq. (57)] along with (61), we can write K in (41) as:

$$K = K_0 \left[1 + \sum_{j=1}^{\infty} k^j \alpha_j \right], \qquad (67)$$

where

$$\alpha_j \equiv \left(\frac{-i}{\hbar}\right)^j \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_1} dt_2 \cdots \int_{t_a}^{t_{j-2}} dt_{j-1} \int_{t_a}^{t_{j-1}} dt_j \,(2\pi\hbar)^{-2j} \int dq_1 \ldots dq_j$$

$$\times\, dp_1 \ldots dp_j \, du_1 \ldots du_j \, dv_1 \ldots dv_j \, F(u_1, v_1, \hbar) \ldots F(u_j, v_j, \hbar)$$

$$\times\, H_1(p_1, q_1, t_1) \ldots H_1(p_j, q_j, t_j)$$

$$\times \exp\left[i \sum_{s=1}^{j} (q_s u_s + p_s v_s + u_s v_s/2)/\hbar \right]$$

$$\times \lim_{t_1 - t_1' \to 0^+} \cdots \lim_{t_j - t_j' \to 0^+} \int_{\mathcal{P}} dw(p,q) \exp\Big\{ (-i/\hbar) \big[u_1 q(t_1)$$

$$+ \cdots + u_j q(t_j) + v_1 p(t_1') + \cdots + v_j p(t_j') \big] \Big\} . \qquad (68)$$

The path integral above is readily recognized as being

$$\mathcal{F}w\left[(-i/\hbar)(u_1 \delta_{t_1} + \ldots + u_j \delta_{t_j}),\ (-i/\hbar)(v_1 \delta_{t_1'} + \cdots + v_j \delta_{t_j'}) \right], \qquad (69)$$

which can be evaluated, by use of (2) and (5)-(9). By virtue of lemma 2, the times entering the integral are now ordered ($t_b \geq t_1 \geq t_2 \geq \ldots \geq t_j \geq t_a$).

The result is:

$$\alpha_j = \left(-\frac{i}{\hbar}\right)^j \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_1} dt_2 \cdots \int_{t_a}^{t_{j-2}} dt_{j-1} \int_{t_a}^{t_{j-1}} dt_j \,(2\pi\hbar)^{-2j}$$

$$\times \int dq_1 \ldots dq_j \, dp_1 \ldots dp_j \, du_1 \ldots du_j \, dv_1 \ldots dv_j$$

$$\times\, F(u_1, v_1, \hbar) \ldots F(u_j, v_j, \hbar) \, H_1(p_1, q_1, t_1) \ldots H_1(p_j, q_j, t_j) \, E_j \ , \qquad (70)$$

where

$$E_j \equiv \exp\Big\{(i/\hbar)\sum_{s=1}^{j}\left(q_s u_s + p_s v_s + u_s v_s/2\right)$$

$$-(i/\hbar T)\sum_{r=1}^{j}\left[u_r(t_r-t_a)q_b + u_r(t_b-t_r)q_a + m(q_b-q_a)v_r\right]$$

$$-(i/\hbar T)\sum_{\substack{r,s=1\\ r\geq s}}^{j}\left[u_r u_s(t_r-t_a)(t_b-t_s)/m \; - m v_r v_s/2\right]$$

$$-(i/\hbar T)\Big[\sum_{\substack{r,s=1\\ r\leq s}}^{j} u_r v_s(t_b-t_r) - \sum_{\substack{s=1,r=2\\ r>s}}^{j} u_r v_s(t_r-t_a)\Big]\Big\}. \tag{71}$$

We must now show that if

$$K_n \equiv K_0\left[1+\sum_{j=1}^{n} k^j \alpha_j\right] \tag{72}$$

satisfies the Schrödinger equation to nth order in k, then K_{n+1} satisfies it to (n+1)th order in k, i.e.

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_b^2} + k\,\underset{\sim}{H}_r - i\hbar\frac{\partial}{\partial t_b}\right)\left[K_0\left(1+\sum_{j=1}^{n+1} k^j\alpha_j\right)\right] = O(k^{n+2}). \tag{73}$$

This can be shown to be true if and only if the coefficient of k^{n+1} is 0, i.e. iff[13]

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial q_b^2} - i\hbar\frac{\partial}{\partial t_b}\right)(K_0\alpha_{n+1}) + \underset{\sim}{H}_1(K_0\alpha_n) = 0 \tag{74}$$

(71) readily reduces to

$$\underset{\sim}{H}_1(K_0\alpha_n) - i\hbar K_0\frac{\partial\alpha_{n+1}}{\partial t_b} - \frac{i\hbar K_0(q_b-q_a)}{T}\frac{\partial\alpha_{n+1}}{\partial q_b}$$
$$-\frac{\hbar^2}{2m}K_0\frac{\partial^2\alpha_{n+1}}{\partial q_b^2} = 0. \tag{75}$$

We will calculate each term in (75) separately and show that they cancel each other out. Using (33) to represent $\underset{\sim}{H}_1$, as we did earlier, we have:

$$\underset{\sim}{H}_1(K_0\alpha_n) = (2\pi\hbar)^{-2}K_0\int dp\,dq\,du\,dv\,F(u,v,\hbar)H_1(p,q,t_b)$$
$$\times\exp\left\{(i/\hbar)\left[uq+pv+uv/2-uq_b-mv^2/2T-mv(q_b-q_a)/T\right]\right\}$$
$$\times(-i/\hbar)^n\int_{t_a}^{t_b}dt_1\int_{t_a}^{t_1}dt_2\cdots\int_{t_a}^{t_{n-1}}dt_n\,(2\pi\hbar)^{-2}\int dq_1\ldots dq_n\,dp_1\ldots dp_n$$
$$\times du_1\ldots du_n\,dv_1\ldots dv_n\,F(u_1,v_1,\hbar)\ldots F(u_n,v_n,\hbar)\,H_1(p_1,q_1,t_1)$$
$$\times\cdots H_1(p_n,q_n,t_n)\exp\left\{(i/\hbar)\sum_{s=1}^{n}(q_su_s+p_sv_s+u_sv_s/2)\right.$$
$$-(i/\hbar T)\sum_{r=1}^{n}\left[u_r(t_r-t_a)(q_b-v)+u_r(t_b-t_r)q_a+m(q_b-v-q_a)v_r\right]$$
$$-(i/\hbar T)\sum_{\substack{r,s=1\\ r\geq s}}^{n}\left[u_ru_s(t_r-t_a)(t_b-t_s)/m-mv_rv_s/2\right]$$
$$\left.-(i/\hbar T)\left[\sum_{\substack{r,s=1\\ r\leq s}}^{n}u_rv_s(t_b-t_r)-\sum_{\substack{s=1,r=2\\ r>s}}^{n}u_rv_s(t_r-t_a)\right]\right\}. \tag{76}$$

The time-derivative term in (75), $-i\hbar K_0\, \partial\alpha_{n+1}/\partial t_b$, can be written as A + B, where B is the derivative of the integrand and A evaluates the integrand at t_b. Thus,

$$
\begin{aligned}
A = & -i\hbar K_0 (-i/\hbar)^{n+1} \int_{t_a}^{t_b} dt_2 \cdots \int_{t_a}^{t_n} dt_{n+1}\, (2\pi\hbar)^{-2(n+1)} \\
& \times \int dq_1 \ldots dq_{n+1}\, dp_1 \ldots dp_{n+1}\, du_1 \ldots du_{n+1}\, dv_1 \ldots dv_{n+1} \\
& \times F(u_1, v_1, \hbar) \ldots F(u_{n+1}, v_{n+1}, \hbar) \\
& \times H_1(p_1, q_1, t_b) H_1(p_2, q_2, t_2) \cdots H_1(p_{n+1}, q_{n+1}, t_{n+1}) \\
& \times \exp\Big\{ (i/\hbar) \sum_{s=1}^{n+1} (q_s u_s + p_s v_s + u_s v_s/2) \\
& - (i/\hbar T) \Big[u_1 T q_b + m(q_b - q_a) v_1 + \sum_{r=2}^{n+1} [u_r (t_r - t_a) q_b + u_r (t_b - t_r) q_a \\
& \qquad + m(q_b - q_a) v_r]\Big] \\
& - (i/\hbar T) \Big[-m v_1^2/2 - m v_1 \sum_{r=2}^{n+1} v_r + \sum_{\substack{r,s=2 \\ r \geq s}}^{n+1} [u_r u_s (t_r - t_a)(t_b - t_s)/m \\
& \qquad - m v_r v_s/2]\Big] \\
& - (i/\hbar T) \Big[\sum_{\substack{r,s=2 \\ r \leq s}}^{n+1} u_r v_s (t_b - t_r) - v_1 \sum_{r=2}^{n+1} u_r (t_r - t_a) \\
& \qquad - \sum_{\substack{s=2, r=3 \\ r > s}}^{n+1} u_r v_s (t_r - t_a) \Big] \Big\} . \qquad (77)
\end{aligned}
$$

The t_b dependence of the integrand of $K_0 \alpha_{n+1}$ is of the form $\exp[at_b/(t_b-t_a)]$. Therefore,

$$B = -i\hbar K_0 (-i/\hbar)^{n+1} \int_{t_a}^{t_b} dt_1 \ldots \int_{t_a}^{t_b} dt_{n+1} \, (2\pi\hbar)^{-2(n+1)}$$

$$\times \int dq_1 \ldots dq_{n+1} \, dp_1 \ldots dp_{n+1} \, du_1 \ldots du_{n+1} \, dv_1 \ldots dv_{n+1}$$

$$\times F(u_1, v_1, \hbar) \ldots F(u_{n+1}, v_{n+1}, \hbar)$$

$$\times H_1(p_1, t_1, t_1) \ldots H_1(p_{n+1}, q_{n+1}, t_{n+1}) \, E_{n+1}$$

$$\times (-i/\hbar T^2) \Big\{ T q_a \sum_{r=1}^{n+1} u_r + \sum_{\substack{r,s=1 \\ r \geq s}}^{n+1} u_r u_s (t_r - t_a) T/m$$

$$+ T \sum_{\substack{r,s=1 \\ r \leq s}}^{n+1} u_r v_s - \sum_{r=1}^{n+1} \big[u_r (t_r - t_a) q_b + u_r (t_b - t_r) q_a$$

$$+ m(q_b - q_a) v_r \big] - \sum_{\substack{r,s=1 \\ r \geq s}}^{n+1} \big[u_r u_s (t_r - t_a)(t_b - t_s)/m - m v_r v_s / 2 \big]$$

$$- \sum_{\substack{r,s=1 \\ r \leq s}}^{n+1} u_r v_s (t_b - t_r) + \sum_{\substack{s=1,\, r=2 \\ r > s}}^{n+1} u_r v_s (t_r - t_a) \Big\} . \qquad (78)$$

The q_b dependence of the integrand of α_{n+1} is of the simple form $\exp(aq_b)$. Therefore,

$$-\frac{i\hbar}{T}K_0(q_b-q_a)\frac{\partial\alpha_{n+1}}{\partial q_b}-\frac{\hbar^2}{2m}K_0\frac{\partial^2\alpha_{n+1}}{\partial q_b^2}$$

$$=i\hbar K_0(-i/\hbar)^{n+1}\int_{t_a}^{t_b}dt_1\cdots\int_{t_a}^{t_n}dt_{n+1}\,(2\pi\hbar)^{-2(n+1)}$$

$$\times\int dq_1\cdots dq_{n+1}\,dp_1\cdots dp_{n+1}\,du_1\cdots du_{n+1}\,dv_1\cdots dv_{n+1}$$

$$\times F(u_1,v_1,\hbar)\cdots F(u_{n+1},v_{n+1},\hbar)$$

$$\times H_1(p_1,q_1,t_1)\cdots H_1(p_{n+1},q_{n+1},t_{n+1})\,E_{n+1}$$

$$\times(-i/\hbar T^2)\Big\{-(q_b-q_a)\sum_{r=1}^{n+1}\left[u_r(t_r-t_a)+mv_r\right]$$

$$+\frac{1}{2m}\Big[\sum_{r=1}^{n+1}\big(u_r(t_r-t_a)+mv_r\big)\Big]^2\Big\}, \qquad (79)$$

where E_{n+1} is defined in (71).

From (76) and (77) one can show that $A+\underset{\sim}{H_1}(K_0\alpha_n)=0$. For this, it is sufficient to make the following changes of variable: in (77), $t_i=t_{i-1}$ for i = 2 to n+1; $c_1=c_{n+1}$ and $c_i=c_{i-1}$ for i = 2 to n+1, where c denotes u, v, p, or q; in (76), $c=c_{n+1}$, where c denotes u, v, p, or q. It is also seen, by rearranging the terms inside the curly brackets of (78), that B added to either side of (79) gives zero.

The boundary condition (42b) is satisfied, as can be seen from the expression (43) for K. Indeed, K_0 satisfies (42b) and $\lim_{t_b \to t_a} \alpha_j = 0$, as can be seen from (71).

This completes the proof of the theorem. Q.E.D.

V. CONCLUSION

The object of this paper was to show that <u>any</u> Hamiltonian operator is amenable to a path-integral treatment, by providing an unambiguous, computationally viable formalism and taking proper account of the correspondence rule leading from the classical function to the quantum operator. The manner in which the correspondence rule is taken into account in a semiclassical expansion of the propagator (in powers of $\hbar$)[14] will be the subject of a follow-up paper.

FOOTNOTES

1. Maurice M. Mizrahi, "The Weyl Correspondence and Path Integrals", J. Math. Phys. 16(1975), 2201-6.

2. L. Cohen, "Correspondence Rules and Path Integrals", J. Math. Phys. 17 (1976), 597-8.

3. J. S. Dowker, "Path Integrals and Ordering Rules", J. Math. Phys. 17(1976), 1873-4.

4. Maurice M. Mizrahi, "Phase Space Path Integrals, Without Limiting Procedure", J. Math. Phys. 19(1978), 298-307.

5. L. Cohen, J. Math. Phys. 7 (1966), 781- .

6. However, if one is interested only in a perturbation series ih α of the propagator corresponding to $\underset{\sim}{H} = [g(t)\underset{\sim}{P}^2/2m + f(t)\underset{\sim}{Q}^2/2 + k(t)(\underset{\sim}{P}\underset{\sim}{Q} + \underset{\sim}{Q}\underset{\sim}{P})/2 + \alpha \underset{\sim}{H}_1(\underset{\sim}{P}, \underset{\sim}{Q}, t)]$, then the method described here applies directly.

7. A similar formalism obtained by different methods can be found in C. DeWitt-Morette, A. Maheshwari, and B. Nelson, "Path Integration in Phase Space", Gen. Rel. and Grav. 8(1977), 581-93.

8. Maurice M. Mizrahi, "On Path Integral Solutions of the Schrödinger Equation, Without Limiting Procedure", J. Math. Phys. 17(1976), 566-75.

9. Maurice M. Mizrahi, "Generalized Hermite Polynomials", J. Comp. and Appl. Math. 1(1975), 273-7.

10. The proof of (19) is similar to the one given in Ref. 8.

11. This scheme for correspondence rules was also used in M. M. Mizrahi, "On the Semiclassical Expansion in Quantum Mechanics for Arbitrary Hamiltonians", J. Math. Phys. 18(1977), 786-90, to determine the range of validity of a commonly-used formula for the WKB approximation of the propagator $\langle q_b, t_b | q_a, t_a \rangle$.

12. This formula can be proved by left-multiplying both sides of (28) by $\exp\left[i(\underset{\sim}{Q}u' + \underset{\sim}{P}v')/\hbar\right]$, taking traces [Note: $\mathrm{tr}\,\underset{\sim}{A} \equiv \int dq \langle q|\underset{\sim}{A}|q\rangle = \int dp \langle p|\underset{\sim}{A}|p\rangle$], and inserting complete sets of states after using Eq. (32).

13. In the k-expansion of the left side of (73), the vanishing of the 0th order term results from the Schrödinger equation for K_0, the vanishing of the coefficient of k results from (65), and the vanishing of the coefficients of k^2 to k^{n+1} is what (74) (from $n = 2$ to $n+1$) proves.

14. As discussed in Ref. 4, section IVB, a semiclassical expansion of a propagator which allows such an expansion [see Ref. 11] yields:

$$K = K_{WKB} \int_{\mathcal{P}_0} dw(p_0, q_0) \exp\{(i/\hbar)\Omega[p_0, q_0]\}$$

where K_{WKB} is the semiclassical approximation to K, $\mathcal{P}_0$ is the same as $\mathcal{P}$ except that $q_a = q_b = 0$, $(p_0, q_0) \in \mathcal{P}_0$,

$$\Omega[p_0, q_0] \equiv -\sum_{n=3}^{\infty} \sum_{k=0}^{n} \frac{n!}{(n-k)!\,k!} \int_T dt \left[\frac{\partial^n H_c}{\partial p^k \partial q^{n-k}}\right]_{\substack{q=q_c \\ p=p_c}} (t)\, p_0^k(t)\, q_0^{n-k}(t),$$

and w absorbs the full quadratic part of the expansion of the action functional about the classical path (q_c, p_c). However, if one expands the exponential and attempts to carry out the path integral before the time integral, the indefiniteness discussed in this paper appears again. One conjectured answer, which remains to be verified, is that the $p_0^k(t)\, q_0^{n-k}(t)$ term should be "time-ordered" (in the manner discussed in this paper) with the same correspondence rule as the original Hamiltonian operator of the problem.

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